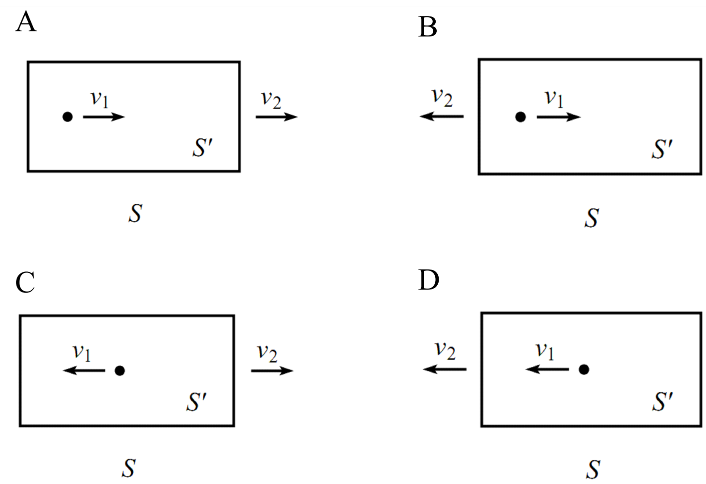
Lecture 4 - Velocity Addition

A Puzzle...

In each of the following four scenarios, v_2 is the speed of frame S' with respect to frame S, and within S' a ball is moving with speed v_1. What is the speed of the ball with respect to frame S in each case?

Out[ ]=



<u>Solution</u>

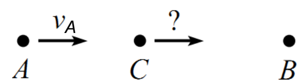
- Part A: This is the velocity addition formula we discussed last time, $u_A = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}$
- Part B: In this case we let $v_2 \to -v_2$ in the velocity addition formula, $u_B = \frac{v_1-v_2}{1-\frac{v_1 v_2}{c^2}}$. Since $v_1 v_2 < c^2$, the denominator is positive. Thus $u > 0$ if $v_1 > v_2$ and $u < 0$ if $v_1 < v_2$, as we would also expect from low-speed (non-relativistic) velocity addition.
- Part C: In this case we let $v_1 \to -v_1$ in the velocity addition formula, $u_C = \frac{-v_1+v_2}{1-\frac{v_1 v_2}{c^2}}$. Note that $u_C = -u_B$, as expected.
- Part D: Here we set both $v_1 \to -v_1$ and $v_2 \to -v_2$ in the velocity addition formula, $u_D = \frac{-v_1-v_2}{1+\frac{v_1 v_2}{c^2}}$. Note that $u_D = -u_A$, as expected. □

Time Dilation and Length Contraction Recap

More Equal Speeds

<u>Example</u>

A moves at speed v_A, and B is at rest. At what speed v_C must C travel, so that she sees A and B approaching her at the same rate?

Out[ ]=


Suppose that A and C arrive at B at the same time. In the lab frame (B's frame), what is the ratio of the distances CB and AC? (The answer to this is very nice and clean. In such cases, you should think of a simple, intuitive explanation for the result!)

Solution

Denote A's, B's, and C's speeds by v_A, v_B, and v_C, respectively. Let us boost all of the speeds by v_C to the left to go into C's frame.

Let's begin with the two easy ones. Boosting C's speed to the left will result in no velocity, by construction. Next, since B is at rest, boosting its speed by v_C to the left yields the speed $-v_C$ pointing to the right (or a speed v_C pointing to the left). Finally, boosting A's speed to the left by v_C yields the velocity $\frac{v_A - v_C}{1 - \frac{v_A v_C}{c^2}}$ pointing to the right. In order for A and B to approach C at the same speed from both directions, we must have

$$\frac{v_A - v_C}{1 - \frac{v_A v_C}{c^2}} = v_C \qquad (1)$$

Solving this using *Mathematica*,

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Simplify[Solve[(vA - vC)/(1 - (vA vC)/c^2) == vC, vC], c > 0]
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$$\left\{\left\{vC \to \frac{c\left(c - \sqrt{c^2 - vA^2}\right)}{vA}\right\}, \left\{vC \to \frac{c\left(c + \sqrt{c^2 - vA^2}\right)}{vA}\right\}\right\}$$

Of the two solutions $v_C = \frac{c^2}{v_A}\left(1 \pm \left\{1 - \left(\frac{v_A}{c}\right)^2\right\}^{1/2}\right)$, only the minus sign solution is physical (i.e. less than c), and hence the speed at which C must travel is $v_C = \frac{c^2}{v_A}\left(1 - \left\{1 - \left(\frac{v_A}{c}\right)^2\right\}^{1/2}\right)$.

If A and C arrive at B at the same time (note that the two events - A arriving at B and C arriving at B - occur at the same time *and* place; therefore they occur simultaneously in *all* frames), then the ratio of the distances will equal

$$\begin{aligned}
\frac{CB}{AC} &= \frac{v_C - v_B}{v_A - v_C} \\
&= \frac{\frac{c^2}{v_A}\left(1 - \left\{1 - \left(\frac{v_A}{c}\right)^2\right\}^{1/2}\right)}{v_A - \frac{c^2}{v_A}\left(1 - \left\{1 - \left(\frac{v_A}{c}\right)^2\right\}^{1/2}\right)} \\
&= \frac{1 - \left\{1 - \left(\frac{v_A}{c}\right)^2\right\}^{1/2}}{\left(\frac{v_A}{c}\right)^2 - \left(1 - \left\{1 - \left(\frac{v_A}{c}\right)^2\right\}^{1/2}\right)} \quad \text{(see comment below)} \\
&= \frac{1}{\left\{1 - \left(\frac{v_A}{c}\right)^2\right\}^{1/2}} \\
&= \gamma_A
\end{aligned} \qquad (2)$$

where in the third step we divided by $\left(1 - \left\{1 - \left(\frac{v_A}{c}\right)^2\right\}^{1/2}\right)$ and used the relation

$$\left(1 - \left\{1 - \left(\frac{v_A}{c}\right)^2\right\}^{1/2}\right)\left(1 + \left\{1 - \left(\frac{v_A}{c}\right)^2\right\}^{1/2}\right) = 1 - \left\{1 - \left(\frac{v_A}{c}\right)^2\right\} = \left(\frac{v_A}{c}\right)^2 \qquad (3)$$

while in the last step we defined

$$\gamma_A = \frac{1}{\left\{1 - \left(\frac{v_A}{c}\right)^2\right\}^{1/2}} \qquad (4)$$

to be A's γ factor in B's frame. This implies that C is γ_A as far from B as she is from A. Note that for non-relativistic speeds $v \ll c$, $\gamma_A \approx 1$ and $v_C = \frac{v_A}{2}$ so that C is midway between A and B.

You may (or at least should) be wondering why in the world $\frac{CB}{AC} = \gamma_A$ is such a simple relation. **In physics, getting such clean results demands a correspondingly simple explanation.** Simple answers imply that if we had considered the problem from a different perspective, we should have easily been able to deduce that $\frac{CB}{AC} = \gamma_A$.

Here is one intuitive reason why the value of $\frac{CB}{AC}$ must come out to be the clean result γ_A. Imagine that in C's frame, A and B are carrying identical jousting sticks as they run toward C; by the problem setup it is clear that the tips of both sticks will hit C simultaneously in this frame. Because those two events occur simultaneously at the same point in C's frame, they occur simultaneously in all frames...including B's frame! But in B's frame, B's stick is uncontracted, while A's stick is length-contracted by a factor γ_A. So when the tips of the two sticks touch C simultaneously, this forces A to be closer to C than B is by a factor γ_A, as desired. □

The Triplet Paradox

Example

Consider the following variation of the twin paradox. A, B, and C each have a clock. In A's reference frame, B flies past A with speed v to the right. When B passes A, they both set their clocks to zero. Also, in A's reference frame, C starts far to the right and moves to the left with speed v. When B and C pass each other, C sets his clock to read the same as B's. Finally, when C passes A, they compare the readings on their clocks. At this event, let A's clock read T_A, and let C's clock read T_C. Define $\gamma = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}$.

(a) Working in A's frame, show that $T_C = T_A / \gamma$
(b) Working in B's frame, show again that $T_C = T_A / \gamma$
(c) Working in C's frame, show again that $T_C = T_A / \gamma$

Solution

Part (a): Let the starting distance between A and C at time $t = 0$ be $2\,d$. In A's reference frame, B and C will meet each other a distance d away from clock A, with both of these clocks moving at speed v. B's clock will be running slow by a factor of γ, so it will be showing a time $\frac{d}{v\,\gamma}$ when B and C meet, and transfer this time over to C.

The time it takes for B and C to meet will equal the time it subsequently takes for A to meet C, since both B and C travel at speed v, and clock C is now retracing B's path. Since C is moving at speed v, the time $\frac{d}{v\,\gamma}$ will elapse on clock C between the time it meets clock B and clock A. Therefore, C's clock will ultimately read $T_C = \frac{2\,d}{v\,\gamma}$. Throughout this entire time, A's clock will read the same amount of time, but without the time dilation factor, $T_A = \frac{2\,d}{v}$. Therefore, $T_C = T_A / \gamma$.

Part (b): Now let's looks at things in B's frame, where A moves away from B at velocity v while C chases A at a velocity given by relativistically adding speed v with v. Let B's clock read t_B when he meets C. Then at this time, B hands off the time t_B to C, and B sees A's clock read $\frac{t_B}{\gamma}$.

We must now determine how much additional time elapses on A's clock and C's clock, by the time they meet. From the velocity-addition formula, B sees C flying by to the left at speed $v_2 \equiv \frac{2\,v}{1+\frac{v^2}{c^2}}$. He also sees A fly by to the left at speed v, but A had a head-start of $v\,t_B$ in front of C. Therefore, if $\tilde{t}$ is the time between the meeting of B and C and the meeting of A and C (as viewed from B), then $v\,t_B = (v_2 - v)\,\tilde{t}$. During this time, A's time increases by $\frac{\tilde{t}}{\gamma}$ while C's clock increases by $\frac{\tilde{t}}{\gamma_2}$ where $\gamma_2 = \frac{1}{\left(1-\frac{v_2^2}{c^2}\right)^{1/2}}$. Thus the total time on clock A is

$$\begin{aligned} T_A &= \frac{t_B}{\gamma} + \frac{\tilde{t}}{\gamma} \\ &= \frac{t_B}{\gamma} + t_B \frac{v}{(v_2 - v)\gamma} \\ &= \frac{t_B}{\gamma}\left(1 + \frac{1+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}\right) \\ &= 2\,\gamma\, t_B \end{aligned} \tag{5}$$

The total time on clock C reads (after some algebra)

$$\begin{aligned} T_C &= t_B + \frac{\tilde{t}}{\gamma_2} \\ &= t_B + \tilde{t}\,\frac{1-v^2}{1+v^2} \\ &= t_B + t_B \frac{1+v^2}{1-v^2}\,\frac{1-v^2}{1+v^2} \\ &= 2\,t_B \end{aligned} \tag{6}$$

Therefore, $T_C = T_A/\gamma$.

Part (c): In C's frame, A and B both approach C, but B does so faster, moving at a velocity of $v_2 \equiv \frac{2v}{1+\frac{v^2}{c^2}}$ as found in Part b. Denote the starting distance between B and C to be $\tilde{d} \equiv \frac{2d}{\gamma}$ (the length contraction of the distance discussed in Part a (although since we want the ratio of T_A to T_C this length contraction cancels out)). Then B and C meet at time $\frac{\tilde{d}}{v_2}$ (as measured by a stationary observer in C's reference frame), at which point B's clock reads $\frac{\tilde{d}}{v_2\,\gamma_2}$, which is the time that B passes off to C. It then takes a time $\frac{\tilde{d}}{v} - \frac{\tilde{d}}{v_2}$ for clock A to reach C, which means that C will ultimately read (after some messy algebra, which should simply be done in Mathematica)

$$\begin{aligned} T_C &= \frac{\tilde{d}}{v_2\,\gamma_2} + \frac{\tilde{d}}{v} - \frac{\tilde{d}}{v_2} \\ &= \frac{\tilde{d}}{v}\left(\frac{v + v_2\,\gamma_2 - v\,\gamma_2}{v_2\,\gamma_2}\right) \\ &= \frac{\tilde{d}}{v\,\gamma^2} \end{aligned} \tag{7}$$

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FullSimplify[d/(v γ^2) == (d/(v2 γ2) + d/v - d/v2) /. γ2 -> 1/Sqrt[1 - v2^2/c^2] /. γ → 1/Sqrt[1 - v^2/c^2] /. v2 → (2 v)/(1 + v^2/c^2), Assumptions → 0 < v < c]
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True
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The total time it takes for A to reach C (as measured in C's frame) equals $\frac{\tilde{d}}{v}$, and due to A's speed v the final time that A reads will be

$$T_A = \frac{\tilde{d}}{v\,\gamma} \tag{8}$$

Therefore, $T_C = T_A/\gamma$. □

Moving at the Speed of Light

One of the interesting quirks about the velocity addition formula is that if you start off moving at c in one frame, then you move in c in another frame. This begs some interesting questions, such as what happens if you accelerate a car to the speed of light, and you turn on your headlights. Would the light move at speed c relative to you, would it all pool inside of the headlight, or would something altogether different happen? Michael Stevens has an amazing YouTube video analyzing this very question.